A Unifying Framework for Deciding Synchronizability

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10/03/2022

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Synchronizability problem

Synchronizability problem

For $\mathcal{S}$ a system and $\mathcal{C}$ a class of MSC, do we have

$$L(\mathcal{S}) \subseteq \mathcal{C} \ ?$$

$\mathcal{C}$ can be, for a given k, the set of

- existentially-k-bounded MSCs
- universally-k-bounded MSCs
- k-synchronisable MSCs

The interest of the synchronizability problem

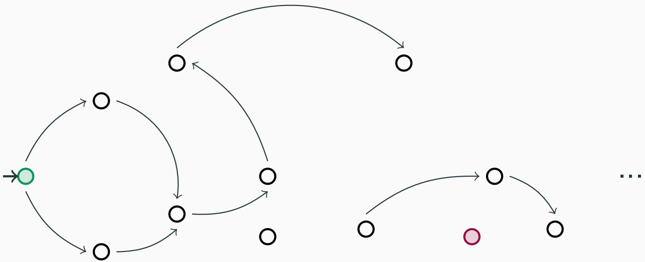
Theorem [Brand and Zafiropoulo, 1983]

Communicating automata systems are Turing-equivalent.

Verification problems are indecidable

so the reachability problem is indecidable.

Reachability problem

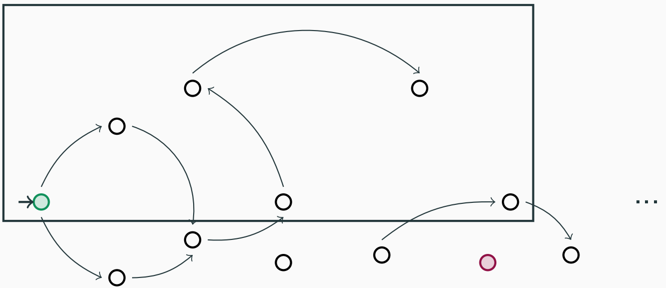


Reachability problem

Is the configuration ○ reachable from the initial configuration ○ ?

Reachability problem

Under-approximation



Under-approximation

Problem applicated to a regular set ⇒

Reachability problem **decidable**

Objective

Membership problem

Does the set of behaviors of the system correspond to the under-approximation ?

Class	pp	mb
$\exists$-k-bounded	Decidable [1]	Decidable [2]
$\forall$-k-bounded	Decidable [1]	Decidable [2]
k-synchronizable	Decidable [3]	Décidable [3]

1. [Genest *et al.*, 2007]
2. [Bollig *et al.*, 2021]
3. [Di Giusto *et al.*, 2020]

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Decidability

In [Bollig *et al.*, 2021], the proof of decidability for mailbox $\exists$ / $\forall$-k-bounded systems uses two concepts :

- the monadic second order logic
- the special treewidth of a graph

Monadic Second Order Logic

Generalities

MSO Logic

- Extension of first-order logic : variables can represent sets.
- Restriction of second-order logic : only quantifications on unary predicates.
- Allows to express properties on graphs

A graph class $\mathcal{C}$ is **MSO-definable** if there exists an MSO-formula φ such that $L(\varphi) = \mathcal{C}$.

Grammar

Grammar :

$$\varphi := x \rightarrow y \mid x \lhd y \mid \lambda(x) = a \mid x = y \mid$$
$$x \in X \mid \exists x.\varphi \mid \exists X.\varphi \mid \varphi \wedge \varphi \mid \neg\varphi$$

When graphs are MSCs we have :

x, y events, a action, (first-order variables)

X set of events, (second-order variables)

φ formula ,

$\lhd$ relation between two matched actions,

$\rightarrow$ order over events of the same process

Example

A matched action :

$matched(x) = \exists y, x \lhd y$

Mailbox order :

$$x \sqsubset y = \bigvee_{q \in \mathbb{P}, a, b \in Send(\_, q, \_)} \lambda(x) = a \land \lambda(y) = b \land$$
$$((matched(x) \land \neg matched(y)) \lor$$
$$\exists x', \exists y'(x \lhd x' \land y \lhd y' \land x' \rightarrow^{+} y'))$$

Special treewidth

Generalities

Intuition

Specify how far a graph is from being a tree.

- Trees have a special treewidth of 1.
- A graph with a treewidth of k can be described by a k-STT (Special Tree Term).
- If a graphs set has a bounded special treewidth, the satisfiability of an MSO-formula on this set is decidable.

k-STT

Syntax

$$\tau := i \mid Add_{i,j}\ \tau \mid Add_i^a\ \tau \mid Forget_i\ \tau \mid \tau \oplus \tau$$

- color
- add edge between nodes of color i and j
- add label to nodes of color i (message or process)
- forget color i
- union of two term if there is no common color

If an STT uses $k+1$ colors, it is a **k-STT**.

Example of k-STT

Create a new node :

$$Add_i^p \ Add_i^a \ i = (i, a, p)$$

An arbitrary graph :

$$\tau = Add_{1,3} Add_{2,1}(Add_{0,1}((0, a, q) \oplus (1, c, p)) \oplus Add_{2,3}((2, b, p) \oplus (3, d, p)))$$

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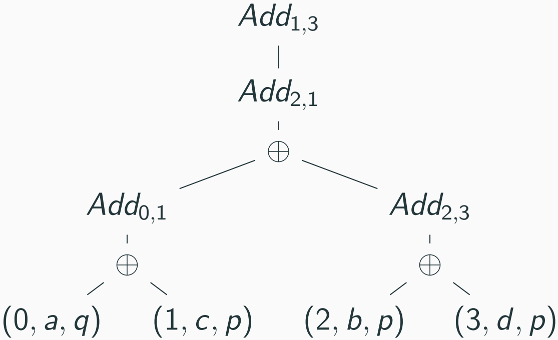

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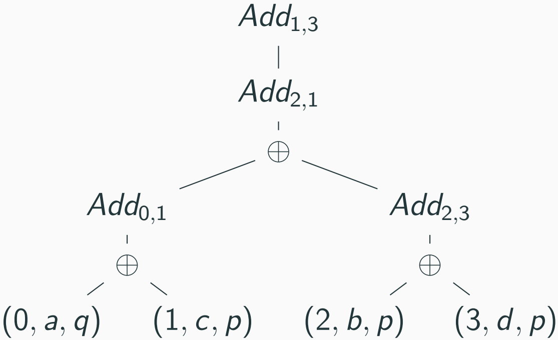
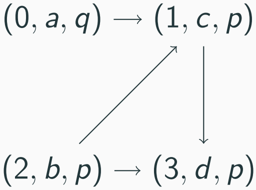
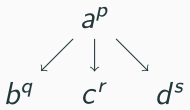
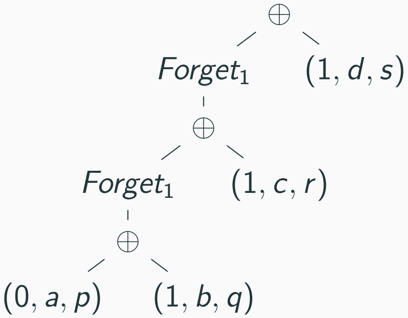




Example

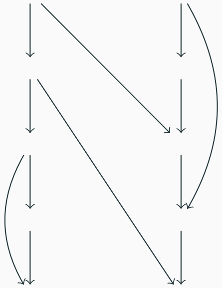
For a tree :





Decomposition game for special treewidth

For a given graph and a given k, we check that the graph has a special treewidth of k. Here, $k = 3$.



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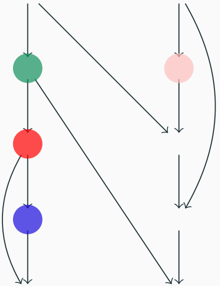

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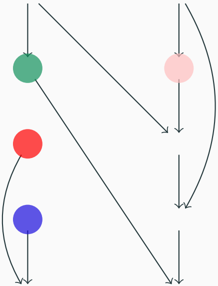

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3) Adam choose the next graph.
4) Eve can put the absent colors on nodes, etc. Until there is no more possible moves.

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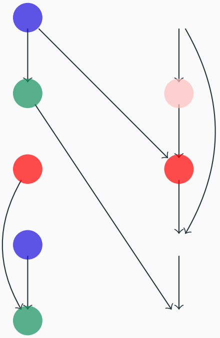

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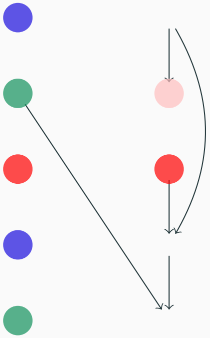

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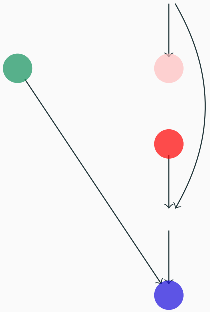

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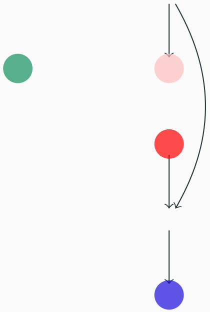

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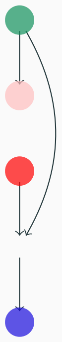

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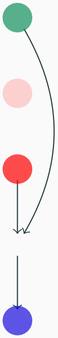

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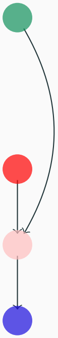

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Decidability proof of synchronizability for existentially bounded systems

Synchronizability problem

Synchronizability problem

For $\mathcal{S}$ a system and $\mathcal{C}$ a class of MSC, do we have

$$L(\mathcal{S}) \subseteq \mathcal{C} \text{ ?}$$

This problem is decidable if :

1. $\mathcal{C}$ is MSO-definable
2. $\mathcal{C}$ is has a k bounded treewidth

Example with $\exists$-B-mailbox bounded MSCs

We choose $\mathcal{C}$ the class of $\exists$-B-mailbox bounded MSCs.

1. Is $\mathcal{C}$ MSO-definable ?
    - We are looking for a formula verifying that there exists a linearization $\exists$-B bounded and runnable with a mailbox communication.
2. Does $\mathcal{C}$ have a k bounded treewidth ?
    - Can we find a bound such that all $\exists$-B-mailbox bounded MSCs have a special treewidth of at most k.

1. Is $\mathcal{C}$ MSO-definable ?

A MSC is $\exists$-B-mailbox bounded if $(\rightarrow \cup \lhd \cup \sqsubset \cup \xrightarrow{rev}_k)^*$ is acyclic.

- we have seen that $\sqsubset$ is MSO-definable
- $\xrightarrow{rev}_k$ makes the link between a reception and the sending of the k-th reception which follows it.
- $r \xrightarrow{rev}_k s = \exists r_1, r_2, \cdots, r_n, r \rightarrow r_1 \rightarrow r_2 \rightarrow \cdots \rightarrow r_n \wedge s \lhd r_n$

So $\mathcal{C}$ is MSO-definable.

2. Does $\mathcal{C}$ have a k bounded treewidth ?

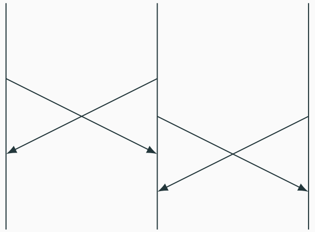
We can prove that for all $\exists$-B-mailbox bounded MSCs in a system, we can bound the special treewidth by

$$k = B \mid \mathbb{P} \mid^2 + \mid \mathbb{P} \mid$$

Eve's strategy for $\mathcal{C}$

A $\exists$-2-mailbox bounded MSC

Let's show that it has a special treewidth of 3
$(3 < 2 \times 3^2 + 3 = 21)$.



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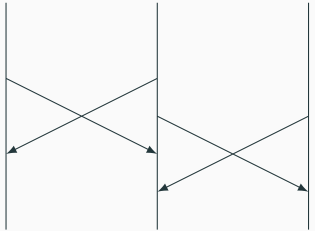

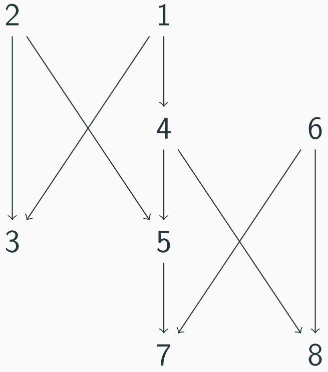

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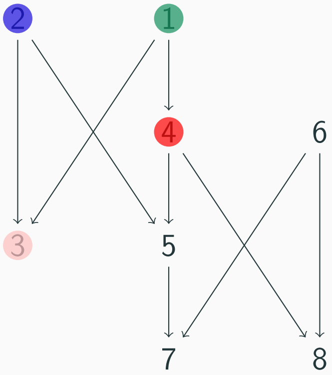

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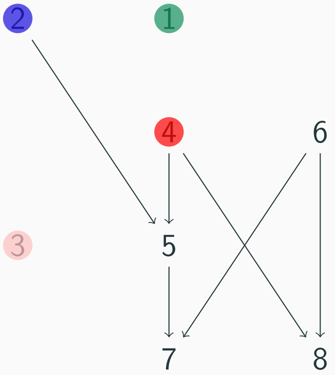

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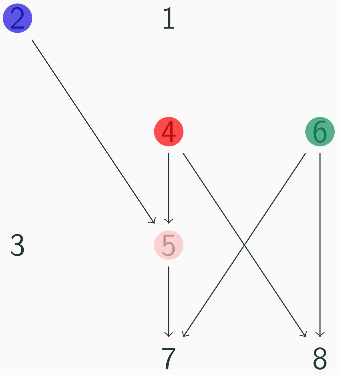

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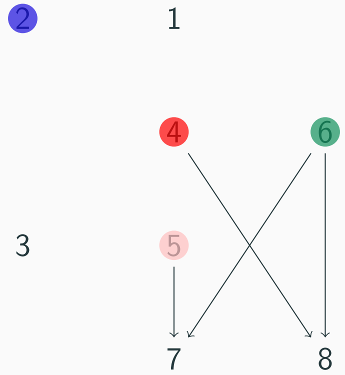

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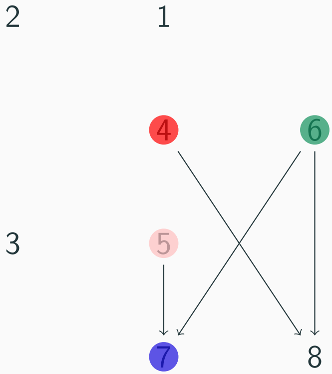

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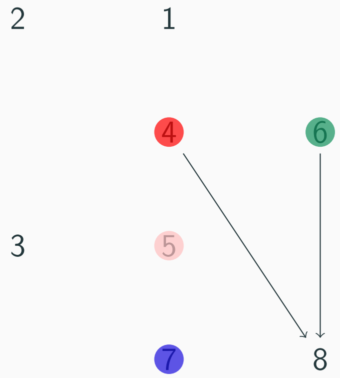

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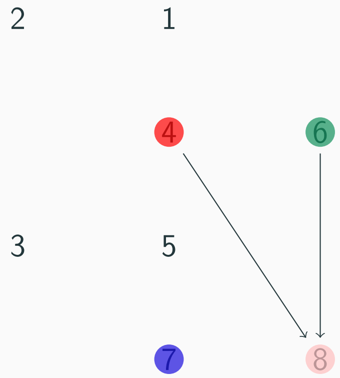

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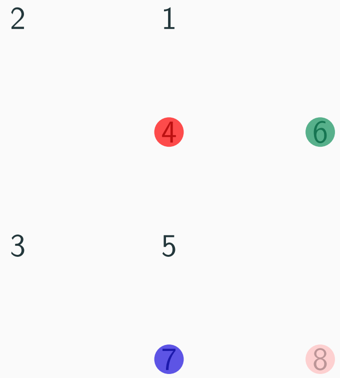

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We chose $\mathcal{C}$ the class of $\exists$-B-mailbox bounded MSCs.

1. Is $\mathcal{C}$ MSO-definable ? $\rightarrow$ Yes
2. Does $\mathcal{C}$ have a k bounded treewidth ? $\rightarrow$ Yes

$\Rightarrow$ The synchronizability problem is decidable for this class.

Conclusion

- Decidability proof for synchronizability problem.
- This method works with different classes, as long as they are MSO-definable and with a bounded special treewidth.
- It has a drawback : if the k is not passed as a parameter, it does not help to find an adequate k
  - "Is there a k such that our system is $\exists$-k-mailbox-bounded ?" is an open problem.